RESOLUTION OF PELLER'S PROBLEM CONCERNING KOPLIENKO-NEIDHARDT TRACE FORMULAE

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Abstract. A formula for the norm of a bilinear Schur multiplier acting from the Cartesian product $\mathcal{S}^2 \times \mathcal{S}^2$ of two copies of the Hilbert-Schmidt classes into the trace class $\mathcal{S}^1$ is established in terms of linear Schur multipliers acting on the space $\mathcal{S}^\infty$ of all compact operators. Using this formula, we resolve Peller's problem on Koplienko-Neidhardt trace formulae. Namely, we prove that there exist a twice continuously differentiable function f with a bounded second derivative, a self-adjoint (unbounded) operator A and a self-adjoint operator $B \in \mathcal{S}^2$ such that

$$f(A+B) - f(A) - \frac{d}{dt}(f(A+tB))\big|_{t=0} \notin \mathcal{S}^1.$$

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $B(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$ equipped with the standard trace Tr. Let $\mathcal{S}^1 = \mathcal{S}^1(\mathcal{H})$ and $\mathcal{S}^2 = \mathcal{S}^2(\mathcal{H})$ be the trace class and the Hilbert-Schmidt class in $B(\mathcal{H})$, respectively.

In 1953, M. G. Krein [16] showed that for a self-adjoint (not necessarily bounded) operator A and a self-adjoint operator $B \in \mathcal{S}^1$ there exists a unique function $\xi \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}(f(A+B) - f(A)) = \int_{\mathbb{R}} f'(t)\xi(t)dt, \tag{1}$$

whenever f is from the Wiener class W_1, that is f is a function on $\mathbb{R}$ with Fourier transform of f' in $L^1(\mathbb{R})$.

The function ξ above is called Lifshitz-Krein spectral shift function and was firstly introduced in a special case by I. M. Lifshitz [17]. It plays an important role in Mathematical Physics and in Scattering Theory, where it appears in the formula of the determinant of scattering matrix (for detailed discussion we refer to [7] and references therein).

Observe that the right-hand side of (1) makes sense for every Lipschitz function f. In 1964 M. G. Krein conjectured that the left-hand side of (1) also makes sense for every Lipschitz function f. More precisely, Krein's conjecture was the following.

Krein's Conjecture. *For any self-adjoint (not necessarily bounded) operator A, for any self-adjoint operator $B \in \mathcal{S}^1$ and for any Lipschitz function f,*

$$f(A+B) - f(A) \in \mathcal{S}^1. \tag{2}$$

The best result concerning the description of the class of functions for which (2) holds is due to V. Peller in [24], who established that (2) holds for f belonging to the Besov class $B^1_{\infty 1}$ (for a definition of this class, see [24] and references therein). However (2) does not hold even for the absolute value function, which is obviously the simplest example of a Lipschitz function (see e.g. [9], [10]). Moreover, there is an example of a continuously differentiable Lipschitz function f and (bounded)

self-adjoint operators A, B with $B \in \mathcal{S}^1$ such that (2) does not hold. The first such example is due to Yu. B. Farforovskaya [12].

Assume now that B is a self-adjoint operator from the Hilbert-Schmidt class $\mathcal{S}^2$. In 1984, L. S. Koplienko, [15], considered the operator

$$f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0}, \tag{3}$$

where by $\frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0}$ we denote the derivative of the map $t \mapsto f(A+tB)$ in the Hilbert-Schmidt norm. He proved that for every fixed self-adjoint operator A there exists a unique function $\eta \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}\Big(f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0}\Big) = \int_{\mathbb{R}} f''(t)\eta(t)dt, \tag{4}$$

if f is an arbitrary rational function with poles off $\mathbb{R}$.

The function η is called Koplienko's spectral shift function (for more information about Koplienko's spectral shift function we refer to [13] and references therein).

It is clear that the right-hand side of (4) makes sense when f is a twice differentiable function with a bounded second derivative. The natural question is then to describe the class of all these functions f such that the left-hand side of (4) is well-defined. Namely, for which function f does the operator (3) belong to $\mathcal{S}^1$? The best result to date is again due to V. Peller [25], who established an affirmative answer under the assumption that f belongs to the Besov class $B^2_{\infty 1}$. In the same paper [25], V. Peller stated the following problem.

Peller's problem. [25, Problem 2] *Suppose that f is a twice continuously differentiable function with a bounded second derivative. Let A be a self-adjoint (possibly unbounded) operator and let B be a self-adjoint operator from $\mathcal{S}^2$. Is it true that*

$$f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0} \in \mathcal{S}^1? \tag{5}$$

In [25, Theorem 4.6], the author defined the operator in (3) for all $f \in B^2_{\infty 1}$ via an approximation process. The precise meaning of (3) in the case of an arbitrary self-adjoint operator A and an arbitrary twice continuously differentiable function f may be a subject of independent investigation, which is beyond the scope of the present paper. However when A is a bounded self-adjoint operator, then the meaning of the operator in (3) is firmly established (see e.g. [4, 5, 6, 20, 21]). From this it is immediate to define uniquely the operator in (3) in the case when A is given by a direct sum $\oplus_{n=1}^{\infty} A_n$, where each A_n is a bounded self-adjoint operator, and $B = \oplus_{n=1}^{\infty} B_n$ is a self-adjoint operator from $\mathcal{S}^2$.

In this paper we answer Peller's question in the negative (see Section 5). More precisely we present a class of twice continuously differentiable functions f with a bounded second derivative and self-adjoint operators $A = \oplus_{n=1}^{\infty} A_n$ and $B = \oplus_{n=1}^{\infty} B_n$ as above, with $B \in \mathcal{S}^2$, such that the operator (3) does not belong to $\mathcal{S}^1$. The operators A_n will be finite rank.

In essence, the construction leading to these counterexamples is finite-dimensional; this construction is presented in Section 4. A key component of our proof is Theorem 6, which provides a new general formula of independent interest for the norm of bilinear Schur multipliers (see Definition 2) from $\mathcal{S}^2 \times \mathcal{S}^2$ into $\mathcal{S}^1$, in terms of a special sequence of Schur multipliers on $\mathcal{S}^\infty$. In Section 3 we establish preliminary results and connect Peller's problem to bilinear Schur multipliers.

2. Bilinear Schur multipliers on $\mathcal{S}^2 \times \mathcal{S}^2$

We regard elements of $B(\ell^2)$ as infinite matrices in the usual way and we let $\|\cdot\|_\infty$ denote the uniform norm on this space. By $\mathcal{S}^p$ we denote the Schatten von Neumann ideal in $B(\ell^2)$ equipped with the Schatten p-norm $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Likewise for any $n \in \mathbb{N}$, we let M_n denote the space of all $n \times n$ matrices with entries in $\mathbb{C}$, equipped with the uniform norm $\|\cdot\|_\infty$, and we use the notation $\mathcal{S}_n^p$ to denote that space equipped with the p-norm $\|\cdot\|_p$.

We let E_{ij} denote the standard matrix units either on $B(\ell^2)$ or on M_n, for $i, j \geq 1$ of for $1 \leq i, j \leq n$.

Let $1 \leq p \leq \infty$. A matrix $M = \{m_{ij}\}_{i,j\geq 1}$ with entries in $\mathbb{C}$ is said to be a *(linear) Schur multiplier on* $\mathcal{S}^p$ if the following action

$$M(A) := \sum_{i,j\geq 1} m_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^p,$$

defines a bounded linear operator on $\mathcal{S}^p$.

Clearly, for the matrix $M = \{m_{ij}\}_{i,j\geq 1}$ to be a linear Schur multiplier on $\mathcal{S}^p$ it is necessary that $\sup_{i,j\geq 1} |m_{ij}| < \infty$. When $p = 2$, this condition is sufficient, that is, a matrix $M = \{m_{ij}\}_{i,j\geq 1}$ is a linear Schur multiplier on $\mathcal{S}^2$ if and only if $\sup_{i,j\geq 1} |m_{ij}| < \infty$. Moreover

$$\|M : \mathcal{S}^2 \to \mathcal{S}^2\| \,=\, \sup_{i,j\geq 1} |m_{ij}|$$

in this case (see e.g. [2, Proposition 2.1]).

A simple duality argument shows that if $1 \leq p, p' \leq \infty$ are conjugate numbers, then a matrix M is a linear Schur multiplier on $\mathcal{S}^p$ if and only if it is a linear Schur multiplier on $\mathcal{S}^{p'}$. Moreover the resulting operators have the same norm, that is, $\|M \colon \mathcal{S}^p \to \mathcal{S}^p\| = \|M \colon \mathcal{S}^{p'} \to \mathcal{S}^{p'}\|$. Linear Schur multipliers on either $\mathcal{S}^1$ or $\mathcal{S}^\infty$ have the following description (see e.g. [27, Theorem 5.1] or [3, Theorem 6.4]).

Theorem 1. *A matrix $M = \{m_{ij}\}_{i,j\geq 1}$ is a linear Schur multiplier on $\mathcal{S}^\infty$ (equivalently, on $\mathcal{S}^1$) if and only if there exist a Hilbert space E and two bounded sequences $(\xi_i)_{i\geq 1}$ and $(\eta_j)_{j\geq 1}$ in E such that*

$$m_{ij} = \langle \xi_i, \eta_j \rangle, \qquad i, j \geq 1. \tag{6}$$

Moreover

$$\|M : \mathcal{S}^\infty \to \mathcal{S}^\infty\| \,=\, \inf\bigl\{\sup_i \|\xi_i\| \, \sup_j \|\eta_j\|\bigr\},$$

where the infimum runs over all possible factorizations (6).

Except for the cases $p = 1, 2, \infty$ mentioned above, there is no known description of linear Schur multipliers on $\mathcal{S}^p$.

The terminology below is adopted from [11], where multilinear Schur products are defined and studied in the context of completely bounded maps.

Definition 2. *Let $1 \leq r \leq \infty$. A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ with entries in $\mathbb{C}$ is said to be a bilinear Schur multiplier into $\mathcal{S}^r$ if the following action*

$$M(A, B) := \sum_{i,j,k\geq 1} m_{ikj} a_{ik} b_{kj} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1}, B = \{b_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2,$$

defines a bounded bilinear operator from $\mathcal{S}^2 \times \mathcal{S}^2$ into $\mathcal{S}^r$.

Of course we can define as well a notion of bilinear Schur multiplier from $\mathcal{S}^p \times \mathcal{S}^q$ into $\mathcal{S}^r$, whenever $1 \leq p, q, r \leq \infty$. The case when $p = q = r = \infty$ is the object of [11]. The main aim of this section is to give a criteria when a matrix M is a bilinear Schur multiplier from $\mathcal{S}^2 \times \mathcal{S}^2$ into $\mathcal{S}^1$ (see Theorems 6, 7, and Corollary 8

below). Before coming to this, we mention another (easier) case which will used in Section 5.

Lemma 3. *A matrix* $M = \{m_{ikj}\}_{i,k,j\geq 1}$ *is a bilinear Schur multiplier into* $\mathcal{S}^2$ *if and only if* $\sup_{i,j,k\geq 1}|m_{ikj}| < \infty$. *Moreover,*

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j,k\geq 1}|m_{ikj}|.$$

Proof. The inequality $\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \leq \sup_{i,j,k\geq 1}|m_{ikj}|$ is achieved by the following computation. Consider $A = \{a_{ik}\}_{i,k\geq 1}$ and $B = \{b_{kj}\}_{k,j\geq 1}$ in $\mathcal{S}^2$. Then applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|M(A,B)\|_2^2 = \Big\| \sum_{i,j,k\geq 1} m_{ikj}a_{ik}b_{kj}E_{ij}\Big\|_2^2 &= \sum_{i,j\geq 1}\Big|\sum_{k\geq 1} m_{ikj}a_{ik}b_{kj}\Big|^2 \\
&\leq \sup_{i,j,k\geq 1}|m_{ikj}|^2 \sum_{i,j\geq 1}\Big(\sum_{k\geq 1}|a_{ik}b_{kj}|\Big)^2 \\
&\leq \sup_{i,j,k\geq 1}|m_{ikj}|^2 \sum_{i,j\geq 1}\sum_{k\geq 1}|a_{ik}|^2\sum_{k\geq 1}|b_{kj}|^2 \\
&\leq \sup_{i,j,k\geq 1}|m_{ikj}|^2\|A\|_2^2\|B\|_2^2.
\end{aligned}$$

The converse inequality is obtained from

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \geq \|M(E_{ik}, E_{kj})\|_2 = |m_{ikj}|,$$

taking the supremum over all $i, j, k \geq 1$. $\square$

We now focus on bilinear Schur multipliers into $\mathcal{S}^1$. We start with some background on tensor products. Given any two Banach spaces X and Y, we let $X \otimes Y$ denote their algebraic tensor product. For every $u \in X \otimes Y$, the projective tensor norm of u is defined as

$$\pi(u) := \inf\Big\{\sum_{i=1}^m \|x_i\|\|y_i\| : \ u = \sum_{i=1}^m x_i \otimes y_i, \ m \in \mathbb{N}\Big\}.$$

Then the completion of $X \otimes Y$ equipped with the norm π is called the projective tensor product of X and Y and is denoted by $X\widehat{\otimes}Y$.

Let Z be another Banach space and let $B_2(X \times Y, Z)$ denote the space of all bounded bilinear operators from $X \times Y$ into Z, equipped with the uniform norm. Next let $B(X\widehat{\otimes}Y, Z)$ denote the Banach space of all bounded linear operators from $X\widehat{\otimes}Y$ into Z, equipped with the uniform norm. Then we have an isometric isomorphism

$$B_2(X \times Y, Z) = B(X\widehat{\otimes}Y, Z), \tag{7}$$

which is given by $T \mapsto \tilde{T}$, where $\tilde{T}(x \otimes y) = T(x, y)$ for any $x \in X$ and $y \in Y$ (see e.g. [29, Theorem 2.9]).

Let $\mathcal{H}$ be a Hilbert space and let $\overline{\mathcal{H}}$ denote its conjugate space. For any h_1, h_2 in $\mathcal{H}$, we may identify $\overline{h_1} \otimes h_2$ with the operator $h \mapsto \langle h, h_1\rangle h_2$ from $\mathcal{H}$ into $\mathcal{H}$. This yields an identification of $\overline{\mathcal{H}} \otimes \mathcal{H}$ with the space of finite rank operators on $\mathcal{H}$, and this identification extends to an isometric isomorphism

$$\overline{\mathcal{H}}\widehat{\otimes}\mathcal{H} \, = \, S^1(\mathcal{H}), \tag{8}$$

see e.g. [22, p. 837].

In the sequel, we regard M_{n^2} as the space of matrices with columns and rows indexed by $\{1, \ldots, n\}^2$. Thus we write $E_{(i,k),(j,l)}$ for its standard matrix units. Then we let $M_n \otimes_{\min} M_n$ denote the minimal tensor product of two copies of

M_n. According to the definition of $\otimes_{\min}$ (see e.g. [31, IV.4.8]), the isomorphism $J_0\colon M_n \otimes_{\min} M_n \to M_{n^2}$ given by

$$J_0(E_{ij} \otimes E_{kl}) = E_{(i,k),(j,l)}, \qquad 1 \le i,j,k,l \le n, \tag{9}$$

is an isometry.

We now give some duality principles. First we recall that ${\mathcal{S}_n^1}^*$ is isometrically isomorphic to M_n through the duality pairing

$$\mathcal{S}_n^1 \times M_n \to \mathbb{C}, \qquad (A,B) \mapsto \operatorname{Tr}\bigl({}^t\!AB\bigr). \tag{10}$$

With this convention (note the use of transposition), the dual basis of $(E_{ij})_{1\le i,j\le n}$ is $(E_{ij})_{1\le i,j\le n}$ itself.

Next we let γ be the cross norm on $\mathcal{S}_n^1 \otimes \mathcal{S}_n^1$ such that

$$\bigl(\mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1\bigr)^* = M_n \otimes_{\min} M_n, \tag{11}$$

through the duality pairing (10) applied twice. More explicitly, for any family $(t_{ijkl})_{1\le i,j,k,l\le n}$ of complex numbers, we have

$$\gamma\Biggl(\sum_{i,j,k,l=1}^n t_{ijkl} E_{ij} \otimes E_{kl}\Biggr) = \sup\Biggl\{\Bigl|\sum_{i,j,k,l=1}^n t_{ijkl}s_{ijkl}\Bigr| \,:\, \Bigl\|\sum_{i,j,k,l=1}^n s_{ijkl}E_{ij}\otimes E_{kl}\Bigr\|_{M_n\otimes_{\min}M_n} \le 1\Biggr\}.$$

Lemma 4. *The isomorphism $J\colon \mathcal{S}_n^2\widehat{\otimes}\mathcal{S}_n^2 \to \mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1$ given by*

$$J(E_{ik}\otimes E_{jl}) = E_{ij}\otimes E_{kl}, \qquad 1\le i,j,k,l\le n,$$

is an isometry.

Proof. According to the equality

$$\Bigl\|\sum_{i,k} c_{ik}E_{ik}\Bigr\|_2 = \Bigl(\sum_{i,k}|c_{ik}|^2\Bigr)^{\frac12}, \qquad c_{ik}\in\mathbb{C},$$

we can naturally identify $\mathcal{S}_n^2$ with either $\ell_{n^2}^2$ or its conjugate space. Then applying the identity (8) with $\mathcal{H}=\ell_{n^2}^2$, we obtain that the mapping $J_1\colon \mathcal{S}_n^2\widehat{\otimes}\mathcal{S}_n^2 \to \mathcal{S}_{n^2}^1$ given by

$$J_1(E_{ik}\otimes E_{jl}) = E_{(i,k),(j,l)}, \qquad 1\le i,j,k,l\le n,$$

is an isometry.

Now let $J_2\colon \mathcal{S}_n^1\otimes_\gamma\mathcal{S}_n^1 \to \mathcal{S}_{n^2}^1$ be the isomorphism given by

$$J_2(E_{ij}\otimes E_{kl}) = E_{(i,k),(j,l)}, \qquad 1\le i,j,k,l\le n.$$

Taking into account the identity (11), we see that J_2^{-1} is the adjoint of J_0. Consequently, J_2^{-1} is an isometry. Since $J = J_2^{-1}J_1$, we deduce that J is an isometry as well. $\square$

We will work with the subspace of $M_n\otimes_{\min}M_n$ spanned by the $E_{rk}\otimes E_{ks}$, for $1\le r,k,s\le n$. The next lemma provides a description of this subspace. We let $(e_1,\ldots,e_n)$ denote the standard basis of ℓ_n^∞.

Lemma 5. *The linear mapping $\theta\colon \ell_n^\infty(M_n) \to M_n\otimes_{\min}M_n$ such that*

$$\theta(e_k\otimes E_{rs}) = E_{rk}\otimes E_{ks}, \qquad 1\le k,r,s\le n,$$

is an isometry.

Proof. Take $y = \sum_{k=1}^n e_k \otimes y_k \in \ell_n^\infty(M_n)$, where $y_k = \sum_{r,s=1}^n y_k(r,s)E_{rs}$. From the definition of θ we have

$$\theta(y) = \sum_{r,s,k=1}^n y_k(r,q)E_{rk} \otimes E_{ks}.$$

Recall the isometric isomorphism J_0 given by (9). Then

$$J_0\theta(y) = \sum_{r,s,k=1}^n y_k(r,s)E_{(r,k),(k,s)}.$$

Let $a = \{a_{rk}\}_{r,k=1}^n, b = \{b_{ls}\}_{l,s=1}^n \in \ell_{n^2}^2$. Then we have

$$\big\langle J_0\theta(y)b, a\big\rangle = \sum_{r,s,k=1}^n y_k(r,s)\langle E_{(r,k),(k,s)}(b), a\rangle = \sum_{r,s,k=1}^n y_k(r,s)a_{rk}b_{ks}.$$

Therefore, using Cauchy-Schwarz, we obtain

$$\begin{aligned}
\big|\big\langle J_0\theta(y)b, a\big\rangle\big| &\leq \sum_{k=1}^n \Big| \sum_{r,s=1}^n y_k(r,s)a_{rk}b_{ks}\Big| \\
&\leq \sum_{k=1}^n \|y_k\| \Big(\sum_{r=1}^n |a_{rk}|^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^n |b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n} \|y_k\| \sum_{k=1}^n \Big(\sum_{r=1}^n |a_{rk}|^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^n |b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n} \|y_k\| \Big(\sum_{k,r=1}^n |a_{rk}|^2\Big)^{\frac{1}{2}} \Big(\sum_{k,s=1}^n |b_{ks}|^2\Big)^{\frac{1}{2}} \\
&\leq \max_{1\leq k\leq n} \|y_k\| \|a\|_2 \|b\|_2.
\end{aligned}$$

It follows that $\|\theta(y)\| \leq \max_{1\leq k\leq n} \|y_k\|$.

Now fix $1 \leq k_0 \leq n$. Take arbitrary $\alpha = \{\alpha_r\}_{r=1}^n$ and $\beta = \{\beta_s\}_{s=1}^n$ in ℓ_n^2. Then define

$$a_{rk} := \begin{cases} \alpha_r, & \text{if } k = k_0 \\ 0 & \text{otherwise} \end{cases}, \qquad b_{ls} := \begin{cases} \beta_s, & \text{if } l = k_0 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\big\langle J_0\theta(y)b, a\big\rangle = \langle y_{k_0}(\beta), \alpha\rangle$$

and moreover, $\|a\|_2 = \|\alpha\|_2$, $\|b\|_2 = \|\beta\|_2$. Therefore, we have $\|y_{k_0}\| \leq \|\theta(y)\|$. Hence, $\|\theta(y)\| \geq \max_{1\leq k\leq n} \|y_k\|$. □

The following theorem is the main result of this section.

Theorem 6. *Let $n \in \mathbb{N}$. Let $M = \{m_{ikj}\}_{i,k,j=1}^n$ be a three-dimensional matrix. For any $1 \leq k \leq n$, let $M(k)$ be the (classical) matrix given by $M(k) = \{m_{ikj}\}_{i,j=1}^n$. Then*

$$\big\|M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\big\| = \sup_{1\leq k\leq n} \big\|M(k) : M_n \to M_n\big\|.$$

Proof. According to the isometric identity (7), the bilinear map $M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1$ induces a linear map $\widetilde{M}\colon \mathcal{S}_n^2\widehat{\otimes}\mathcal{S}_n^2 \to \mathcal{S}_n^1$ with $\|M\| = \|\widetilde{M}\|$. Consider

$$T_M = (\widetilde{M}J^{-1})^*\colon M_n \to M_n \otimes_{\min} M_n,$$

where J is given by Lemma 4. The latter implies that

$$\|T_M\| = \big\|M\colon \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\big\|. \tag{12}$$

For any $1 \le r, s \le n$, we have

$$\begin{aligned}\big\langle T_M(E_{rs}), E_{ij} \otimes E_{kl}\big\rangle &= \big\langle E_{rs}, \widetilde{M} J^{-1}(E_{ij} \otimes E_{kl})\big\rangle \\ &= \big\langle E_{rs}, \widetilde{M}(E_{ik} \otimes E_{jl})\big\rangle \\ &= \begin{cases} m_{ikl}\langle E_{rs}, E_{il}\rangle, & \text{if } \ k = j \\ \quad 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} m_{ikl}, & \text{if } \ k = j, \ r = i, \ s = l \\ \ 0 & \text{otherwise} \end{cases},\end{aligned}$$

for all $1 \le i, j, k, l \le n$. Hence

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks} E_{rk} \otimes E_{ks}.$$

This shows that T_M maps into the range of the operator θ introduced in Lemma 5 and that

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks}\, \theta(e_k \otimes E_{rs}).$$

By linearity this implies that for any $C \in M_n$,

$$T_M(C) = \theta\Big(\sum_{k=1}^{n} e_k \otimes [M(k)](C)\Big).$$

Appyling Lemma 5, we deduce that

$$\|T_M(C)\| = \max_k \big\|[M(k)](C)\big\|, \qquad C \in M_n.$$

From this identity we obtain that $\|T_M\| = \max_k \|M(k)\|$. Combining with (12) we obtain the desired identity $\|M\| = \max_k \|M(k)\|$. ☐

For the sake of completeness we give an infinite dimensional version of the previous theorem.

Theorem 7. *A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\ge 1}$ is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if the matrix $M(k) = \{m_{ikj}\}_{i,j\ge 1}$ is a linear Schur multiplier on $\mathcal{S}^\infty$ for every $k \ge 1$ and $\sup_{k\ge 1} \|M(k) : \mathcal{S}^\infty \to \mathcal{S}^\infty\| < \infty$. Moreover,*

$$\big\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| = \sup_{k\ge 1} \big\|M(k) : \mathcal{S}^\infty \to \mathcal{S}^\infty\big\|$$

in this case.

Proof. Consider a three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\ge 1}$ and set $M(k) = \{m_{ikj}\}_{i,j\ge 1}$. For any $n \ge 1$, let

$$M_{(n)} = \{m_{ikj}\}_{1\le i,j\le n} \quad \text{and} \quad M_{(n)}(k) = \{m_{ikj}\}_{1\le i,k,j\le n}$$

be the standard truncations of these matrices.

We may identify $\mathcal{S}^2_n$ (respectively $\mathcal{S}^\infty_n$) with the subspace of $\mathcal{S}^2$ (respectively $\mathcal{S}^\infty$) spanned by $\{E_{ij} : 1 \le i, j \le n\}$. Then the union $\cup_{n\ge 1}\mathcal{S}^2_n$ is dense in $\mathcal{S}^2$. Hence by a standard density argument, M is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if $\sup_{n\ge 1} \|M_{(n)} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\| < \infty$, and in this case

$$\big\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1\big\| \ = \sup_{n\ge 1} \big\|M_{(n)} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\big\|.$$

Likewise $\cup_{n\ge 1}\mathcal{S}^\infty_n$ is dense in the space $\mathcal{S}^\infty$ of all compact operators, for any $k \ge 1$ $M(k)$ is a linear Schur multiplier on $\mathcal{S}^\infty$ if and only if $\sup_{n\ge 1} \|M_{(n)}(k) : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\| < \infty$, and

$$\big\|M(k) : \mathcal{S}^\infty \to \mathcal{S}^\infty\big\| \ = \sup_{n\ge 1}\big\|M_{(n)}(k) : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\big\|.$$

in this case.

Combining the above two approximation results with Theorem 6, we obtain the result. □

Theorem 7 together with Theorem 1 yield the following result.

Corollary 8. *A three-dimensional matrix $M = \{m_{ikj}\}_{i,k,j\geq 1}$ is a bilinear Schur multiplier into $\mathcal{S}^1$ if and only if there exist a Hilbert space E and two bounded families $(\xi_{ik})_{i,k\geq 1}$ and $(\eta_{jk})_{j,k\geq 1}$ in E such that*

$$m_{ikj} = \langle \xi_{ik}, \eta_{jk}\rangle, \qquad i,k,j \geq 1.$$

Moreover

$$\left\| M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^1 \right\| = \inf\bigl\{\sup_{i,k} \|\xi_{ik}\| \sup_{j,k} \|\eta_{jk}\|\bigr\},$$

where the infimum runs over all possible such factorizations.

3. Schur multipliers associated with a function and self-adjoint operators

Throughout this section we work with finite-dimensional operators. We fix an integer $n \geq 1$ and regard $\mathbb{C}^n$ as equipped with its standard Hermitian structure.

Consider two orthonormal bases $e = \{e_j\}_{j=1}^n$ and $e' = \{e'_i\}_{i=1}^n$ in $\mathbb{C}^n$. Then every linear operator $A \in B(\mathbb{C}^n)$ is associated with a matrix $A = \{a_{ij}\}_{i,j=1}^n$, where $a_{ij} = \langle A(e_j), e'_i\rangle$. Sometimes we use the notation $a_{ij}^{e',e}$ instead of a_{ij} to emphasize corresponding bases.

For any unit vector $x \in \mathbb{C}^n$ we let P_x denote the projection on the linear span of x, that is, $P_x(y) = \langle y, x\rangle x$, $y \in \mathbb{C}^n$.

3.1. **Linear Schur multipliers.** Let $A_0, A_1 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators. For $j = 0, 1$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthonormal basis of eigenvectors for A_j, and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the associated n-tuple of eigenvalues, that is, $A_j(\xi_i^{(j)}) = \lambda_i^{(j)} \xi_i^{(j)}$. Without loss of generality, we assume that $\{\lambda_i^{(j)}\}_{i=1}^{n_j}$ is the set of pairwise distinct eigenvalues of the operator A_j, where $n_j \in \mathbb{N}$, $n_j \leq n$. Denote

$$E_i^{(j)} = \sum_{\substack{k=1 \\ \lambda_k^{(j)} = \lambda_i^{(j)}}}^{n} P_{\xi_k^{(j)}}, \quad 1 \leq i \leq n_j, \tag{13}$$

that is, $E_i^{(j)}$ is a spectral projection of the operator A_j associated with the eigenvalue $\lambda_i^{(j)}$.

Let $\phi : \mathbb{R}^2 \to \mathbb{C}$ be a bounded Borel function. Define a linear operator $T_\phi^{A_0,A_1} : B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ given by

$$T_\phi^{A_0,A_1}(X) = \sum_{i,k=1}^{n} \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}}, \quad X \in B(\mathbb{C}^n). \tag{14}$$

Alternatively, when it is more convenient, we will use the representation of $T_\phi^{A_0,A_1}(X)$ in the form

$$T_\phi^{A_0,A_1}(X) = \sum_{i=1}^{n_0} \sum_{k=1}^{n_1} \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) E_i^{(0)} X E_k^{(1)}, \quad X \in B(\mathbb{C}^n). \tag{15}$$

It is not difficult to see that if we identify $B(\mathbb{C}^n)$ with M_n by associating X with the matrix $\{x_{ik}^{\xi_0,\xi_1}\}_{i,k=1}^n$, then the operator $T_\phi^{A_0,A_1}$ acts as a linear Schur multiplier

$\{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^n$. Indeed,

$$\big\langle (P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}})(\xi_s^{(1)}), \xi_r^{(0)} \big\rangle = \begin{cases} \langle X(\xi_s^{(1)}), \xi_r^{(0)} \rangle = x_{rs}^{\xi_0,\xi_1}, & \text{if } \ s = k, \ r = i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\big\langle T_\phi^{A_0,A_1}(X)(\xi_k^{(1)}), \xi_i^{(0)} \big\rangle = \phi(\lambda_i^{(0)}, \lambda_k^{(1)}) x_{ik}^{\xi_0,\xi_1},$$

which implies that $T_\phi^{A_0,A_1} \sim \{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^n \colon M_n \to M_n$. Since these identifications are isometric ones, we deduce that

$$\|T_\phi^{A_0,A_1} \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\| = \big\| \{\phi(\lambda_i^{(0)}, \lambda_k^{(1)})\}_{i,k=1}^n \colon \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty \big\|. \tag{16}$$

The operator $T_\phi^{A_0,A_1}$ is called a linear Schur multiplier associated with ϕ and A_0, A_1.

3.2. **Bilinear Schur multipliers.** Similarly, we introduce bilinear Schur multipliers associated to a triple of self-adjoint operators.

Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be diagonalizable self-adjoint operators and for any $j = 0, 1, 2$, let $\xi_j = \{\xi_i^{(j)}\}_{i=1}^n$ be an orthornomal basis of eigenvectors of A_j and let $\{\lambda_i^{(j)}\}_{i=1}^n$ be the corresponding n-tuple of eigenvalues.

Let $\psi : \mathbb{R}^3 \to \mathbb{C}$ be a bounded Borel function. Define a bilinear operator $T_\psi^{A_0,A_1,A_2} : B(\mathbb{C}^n) \times B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ by setting

$$T_\psi^{A_0,A_1,A_2}(X, Y) = \sum_{i,j,k=1}^n \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}} \tag{17}$$

for any $X, Y \in B(\mathbb{C}^n)$. Assume that $\{\lambda_i^{(j)}\}_{i=1}^{n_j}$ is the set of pairwise distinct eigenvalues of the operator A_j. Then alternatively, using the spectral projections (13), we can write

$$T_\psi^{A_0,A_1,A_2}(X, Y) = \sum_{i=1}^{n_0} \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) E_i^{(0)} X E_k^{(1)} Y E_j^{(2)} \tag{18}$$

for any $X, Y \in B(\mathbb{C}^n)$.

Let us consider two different identifications of $B(\mathbb{C}^n)$ with M_n. On the one hand, we identify X with the matrix $\{x_{ik}^{\xi_0,\xi_1}\}_{i,k=1}^n$, where $x_{ik}^{\xi_0,\xi_1} = \langle X(\xi_k^{(1)}), \xi_i^{(0)} \rangle$. On the other hand we identify Y with $\{y_{kj}^{\xi_1,\xi_2}\}_{k,j=1}^n$, where $y_{kj}^{\xi_1,\xi_2} = \langle Y(\xi_j^{(2)}), \xi_k^{(1)} \rangle$. Under these identifications, the operator $T_\psi^{A_0,A_1,A_2}$ acts as a bilinear Schur multiplier associated with the matrix $M = \{\psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)})\}_{i,j,k=1}^n$. Indeed,

$$\big\langle (P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}})(\xi_s^{(2)}), \xi_r^{(0)} \big\rangle = \big\langle Y(\xi_s^{(2)}), \xi_k^{(1)} \big\rangle \big\langle X(\xi_k^{(1)}), \xi_r^{(0)} \big\rangle = y_{ks}^{\xi_1,\xi_2} x_{rk}^{\xi_0,\xi_1}$$

if $s = j, r = i$, and

$$\big\langle (P_{\xi_i^{(0)}} X P_{\xi_k^{(1)}} Y P_{\xi_j^{(2)}})(\xi_s^{(2)}), \xi_r^{(0)} \big\rangle = 0$$

otherwise.

Therefore,

$$\big\langle T_\psi^{A_0,A_1,A_2}(X, Y)(\xi_s^{(2)}), \xi_r^{(0)} \big\rangle = \sum_{k=1}^n \psi(\lambda_r^{(0)}, \lambda_k^{(1)}, \lambda_s^{(2)}) y_{ks}^{\xi_1,\xi_2} x_{rk}^{\xi_0,\xi_1},$$

which implies

$$T_\psi^{A_0,A_1,A_2}(X, Y) = \sum_{i,j,k=1}^n \psi(\lambda_i^{(0)}, \lambda_k^{(1)}, \lambda_j^{(2)}) x_{ik}^{\xi_0,\xi_1} y_{kj}^{\xi_1,\xi_2} E_{ij}^{\xi_0,\xi_2}.$$

Since these identifications are isometric ones with respect to all Schatten norms, we deduce the formula

$$\big\|T_\psi^{A_0,A_1,A_2}\colon \mathcal{S}_n^2\times\mathcal{S}_n^2\to\mathcal{S}_n^1\big\| = \big\|\{\psi(\lambda_i^{(0)},\lambda_k^{(1)},\lambda_j^{(2)})\}_{i,j,k=1}^n\colon \mathcal{S}_n^2\times\mathcal{S}_n^2\to\mathcal{S}_n^1\big\|. \tag{19}$$

The operator $T_\psi^{A_0,A_1,A_2}$ is called a bilinear Schur multiplier associated with ψ and the operators A_0, A_1, A_2.

Operators $T_\psi^{A_0,A_1,A_2}$ present a special case of what is known in the literature as "multiple operator integrals". We refer to [23, 30, 26, 1, 28] for additional information on this notion.

3.3. **A few properties of Schur multipliers.** In this subsection, $\phi\colon \mathbb{R}^2\to\mathbb{C}$ and $\psi\colon\mathbb{R}^3\to\mathbb{C}$ denote arbitrary bounded Borel functions, and $n\in\mathbb{N}$ is a fixed integer. The following lemma gives some nice properties of bilinear Schur multipliers.

Lemma 9. *Let $A_0, A_1, A_2\in B(\mathbb{C}^n)$ be self-adjoint operators. Let I_n be the identity operator in $B(\mathbb{C}^n)$. Then for $j=0,1$ we have*

(*i*)
$$T_\psi^{A_0,A_1,A_2}(A_j,X) = T_{\psi_j}^{A_0,A_1,A_2}(I_n,X),\quad X\in B(\mathbb{C}^n),$$
where
$$\psi_j(x_0,x_1,x_2) = x_j\psi(x_0,x_1,x_2),\quad x_0,x_1,x_2\in\mathbb{R}.$$

(*ii*)
$$T_\phi^{A_j,A_2}(X) = T_{\tilde\psi_j}^{A_0,A_1,A_2}(I_n,X),\quad X\in B(\mathbb{C}^n),$$
where
$$\tilde\psi_j(x_0,x_1,x_2) = \phi(x_j,x_2),\quad x_0,x_1,x_2\in\mathbb{R}.$$

Proof. Let us prove the assertion for $j=0$ only. The proof for $j=1$ is similar.

(*i*). For $X\in B(\mathbb{C}^n)$ we have

$$\begin{aligned}
T_\psi^{A_0,A_1,A_2}(A_0,X) &= \sum_{i,j,k=1}^n \psi(\lambda_i^{(0)},\lambda_k^{(1)},\lambda_j^{(2)})P_{\xi_i^{(0)}}A_0P_{\xi_k^{(1)}}XP_{\xi_j^{(2)}}\\
&= \sum_{i,j,k=1}^n \psi(\lambda_i^{(0)},\lambda_k^{(1)},\lambda_j^{(2)})P_{\xi_i^{(0)}}\Big(\sum_{r=1}^n\lambda_r^{(0)}P_{\xi_r^{(0)}}\Big)P_{\xi_k^{(1)}}XP_{\xi_j^{(2)}}\\
&= \sum_{i,j,k=1}^n \lambda_i^{(0)}\psi(\lambda_i^{(0)},\lambda_k^{(1)},\lambda_j^{(2)})P_{\xi_i^{(0)}}I_nP_{\xi_k^{(1)}}XP_{\xi_j^{(2)}}\\
&= T_{\psi_0}^{A_0,A_1,A_2}(I_n,X).
\end{aligned}$$

(*ii*). For $X\in B(\mathbb{C}^n)$ we have

$$\begin{aligned}
T_{\tilde\psi_0}^{A_0,A_1,A_2}(I_n,X) &= \sum_{i,j,k=1}^n \tilde\psi_0(\lambda_i^{(0)},\lambda_k^{(1)},\lambda_j^{(2)})P_{\xi_i^{(0)}}I_nP_{\xi_k^{(1)}}XP_{\xi_j^{(2)}}\\
&= \sum_{i,j=1}^n \phi(\lambda_i^{(0)},\lambda_j^{(2)})P_{\xi_i^{(0)}}\Big(\sum_{k=1}^n P_{\xi_k^{(1)}}\Big)XP_{\xi_j^{(2)}}\\
&= \sum_{i,j=1}^n \phi(\lambda_i^{(0)},\lambda_j^{(2)})P_{\xi_i^{(0)}}XP_{\xi_j^{(2)}}\\
&= T_\phi^{A_0,A_2}(X).
\end{aligned}$$

$\square$

Lemma 10. *Let $A \in B(\mathbb{C}^n)$ be a self-adjoint operator and $X, Y \in B(\mathbb{C}^n)$. Let*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}.$$

Then

$$T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X}, \tilde{X}) = \begin{pmatrix} T_\psi^{A,A,A}(X,Y) & 0 \\ 0 & T_\psi^{A,A,A}(Y,X) \end{pmatrix}.$$

Proof. Let $\{\lambda_i\}_{i=1}^m$ be the set of distinct eigenvalues of the operator A, $m \le n$, and let E_i^A be the spectral projection of A associated with λ_i, $1 \le i \le m$. Clearly, the operator $\tilde{A}$ has the same set $\{\lambda_i\}_{i=1}^m$ of distinct eigenvalues and the spectral projection of the operator $\tilde{A}$ associated with λ_i is given by

$$E_i^{\tilde{A}} = \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^A \end{pmatrix}, \quad 1 \le i \le m.$$

Therefore, we have

$$\begin{aligned} T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X}, \tilde{X}) &= \sum_{i,k,j=1}^m \psi(\lambda_i, \lambda_k, \lambda_j) \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^A \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \times \\ &\qquad \begin{pmatrix} E_k^A & 0 \\ 0 & E_k^A \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \begin{pmatrix} E_j^A & 0 \\ 0 & E_j^A \end{pmatrix} \\ &= \sum_{i,k,j=1}^m \psi(\lambda_i, \lambda_k, \lambda_j) \begin{pmatrix} E_i^A X E_k^A Y E_j^A & 0 \\ 0 & E_i^A Y E_k^A X E_j^A \end{pmatrix} \\ &= \begin{pmatrix} T_\psi^{A,A,A}(X,Y) & 0 \\ 0 & T_\psi^{A,A,A}(Y,X) \end{pmatrix}. \end{aligned}$$

□

Lemma 11. *Let $A, B \in B(\mathbb{C}^n)$ be self-adjoint operators with the same set of eigenvalues and $X, Y \in B(\mathbb{C}^n)$. Let*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{Y} = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}.$$

Then

$$T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X}, \tilde{Y}) = \begin{pmatrix} 0 & T_\psi^{A,B,B}(X,Y) \\ 0 & 0 \end{pmatrix}.$$

Proof. Let $\{\lambda_i\}_{i=1}^m$ be the set of distinct eigenvalues of the operator A, $m \le n$, and let E_i^A (resp. E_i^B) be the spectral projection of A (resp. B) associated with λ_i, $1 \le i \le m$. Since A and B have the same set of eigenvalues, the operator $\tilde{A}$ has the same set $\{\lambda_i\}_{i=1}^m$ of distinct eigenvalues and the spectral projection of the operator $\tilde{A}$ associated with λ_i is given by

$$E_i^{\tilde{A}} = \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^B \end{pmatrix}, \quad 1 \le i \le m.$$

Therefore, we have

$$\begin{aligned}
T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X},\tilde{Y}) &= \sum_{i,k,j=1}^{m} \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} E_i^A & 0 \\ 0 & E_i^B \end{pmatrix} \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \times \\
&\qquad\qquad \begin{pmatrix} E_k^A & 0 \\ 0 & E_k^B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} E_j^A & 0 \\ 0 & E_j^B \end{pmatrix} \\
&= \sum_{i,k,j=1}^{m} \psi(\lambda_i,\lambda_k,\lambda_j) \begin{pmatrix} 0 & E_i^A X E_k^B Y E_j^B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & T_\psi^{A,B,B}(X,Y) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

□

Lemma 12. *Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be self-adjoint operators. For any $a \neq 0 \in \mathbb{R}$ we have that*

$$T_\psi^{aA_0,aA_1,aA_2} = T_{\psi_a}^{A_0,A_1,A_2},$$

where

$$\psi_a(x_0,x_1,x_2) = \psi(ax_0,ax_1,ax_2), \quad x_0,x_1,x_2 \in \mathbb{R}.$$

Proof. Let $\{\lambda_i^{(j)}\}_{i=1}^{n_j}$ be the set of distinct eigenvalues of A_j, $j = 0,1,2$. Fix $a \neq 0 \in \mathbb{R}$. It is clear that for any j, $\{a\lambda_i^{(j)}\}_{i=1}^{n_j}$ is the set of distinct eigenvalues of aA_j, and that the corresponding spectral projections coincide, that is, $E_i^{aA_j} = E_i^{A_j}$ for any $i = 1,\ldots,n_j$. Therefore, for $X, Y \in B(\mathbb{C}^n)$, we have

$$\begin{aligned}
T_\psi^{aA_0,aA_1,aA_2}(X,Y) &= \sum_{i=1}^{n_0}\sum_{k=1}^{n_1}\sum_{j=1}^{n_2} \psi\big(a\lambda_i^{(0)}, a\lambda_k^{(1)}, a\lambda_j^{(2)}\big) E_i^{A_0} X E_k^{A_1} Y E_j^{A_2} \\
&= T_{\psi_a}^{A_0,A_1,A_2}(X,Y).
\end{aligned}$$

□

Lemma 13. *Let $A, B \in B(\mathbb{C}^n)$ be self-adjoint operators and let $\{U_m\}_{m\geq 1}$ be a sequence of unitary operators from $B(\mathbb{C}^n)$ such that $U_m \to I_n$ as $m \to \infty$. Let also $X, Y \in B(\mathbb{C}^n)$ and sequences $\{X_m\}_{m\geq 1}$ and $\{Y_m\}_{m\geq 1}$ in $B(\mathbb{C}^n)$ such that $X_m \to X$ and $Y_m \to Y$ as $m \to \infty$. Let $\psi, \psi_m : \mathbb{R}^3 \to \mathbb{C}$ be bounded Borel functions such that $\psi_m \to \psi$ pointwise as $m \to \infty$. Then*

$$T_{\psi_m}^{U_m A U_m^*,B,B}(X_m,Y_m) \longrightarrow T_\psi^{A,B,B}(X,Y), \quad m \to \infty. \tag{20}$$

Proof. Let $\{\lambda_i\}_{i=1}^{m_0}$ and $\{\mu_k\}_{k=1}^{m_1}$ be the set of distinct eigenvalues of the operators A and B, respectively, $m_0, m_1 \leq n$, and let E_i^A (resp. E_k^B) be the spectral projection of A (resp. B) associated with λ_i (resp. μ_k), $1 \leq i \leq m_0$ (resp. $1 \leq k \leq m_1$). It is clear that the sequence $\{\lambda_i\}_{i=1}^{m_0}$ is the sequence of eigenvalues of $U_m A U_m^*$ and that the spectral projection of $U_m A U_m^*$ associated with λ_i is given by

$$E_i^{U_m A U_m^*} = U_m E_i^A U_m^*, \quad 1 \leq i \leq m_0.$$

Observe that

$$\begin{aligned}
T_{\psi_m}^{U_m A U_m^*,B,B}(X_m,Y_m) &= \sum_{i=1}^{m_0}\sum_{j,k=1}^{m_1} \psi_m(\lambda_i,\mu_k,\mu_j) E_i^{U_m A U_m^*} X E_k^B Y E_j^B \\
&= U_m \Big(\sum_{i=1}^{m_0}\sum_{j,k=1}^{m_1} \psi_m(\lambda_i,\mu_k,\mu_j) E_i^A (U_m^* X) E_k^B Y E_j^B \Big) \\
&= U_m T_{\psi_m}^{A,B,B}(U_m^* X, Y).
\end{aligned}$$

We claim that $T_{\psi_m}^{A,B,B}(U_m^* X, Y) \to T_{\psi}^{A,B,B}(X,Y)$. Indeed, we have

$$\begin{aligned}
&\|T_{\psi_m}^{A,B,B}(U_m^* X, Y) - T_{\psi}^{A,B,B}(X,Y)\|_\infty \\
&\le \|T_{\psi_m}^{A,B,B}(U_m^* X, Y) - T_{\psi_m}^{A,B,B}(X,Y)\|_\infty + \|T_{\psi_m}^{A,B,B}(X,Y) - T_{\psi}^{A,B,B}(X,Y)\|_\infty \\
&\le \|T_{\psi_m}^{A,B,B}(U_m^* X - X, Y)\|_\infty + \|T_{\psi_m - \psi}^{A,B,B}(X,Y)\|_\infty \\
&\le \sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} |\psi_m(\lambda_i, \mu_k, \mu_j)| \|U_m X - X\|_\infty \|Y\|_\infty + \\
&\qquad\qquad \sum_{i=1}^{m_0} \sum_{j,k=1}^{m_1} |\psi_m - \psi|(\lambda_i, \mu_k, \mu_j) \|X\|_\infty \|Y\|_\infty .
\end{aligned}$$

This upper bound tends to 0 as $m \to \infty$, which proves the claim.

Now since $U_m \to I_n$, we have

$$U_m T_{\psi_m}^{A,B,B}(U_m^* X, Y) - T_{\psi_m}^{A,B,B}(U_m^* X, Y) \longrightarrow 0$$

as $m \to \infty$. The result follows at once. □

Lemma 14. *Let $A \in B(\mathbb{C}^n)$ be a self-adjoint operator and let $X \in B(\mathbb{C}^n)$ commute with A. Let $\widehat{\psi} \colon \mathbb{R} \to \mathbb{R}$ be defined by $\widehat{\psi}(x) = \psi(x,x,x)$, $x \in \mathbb{R}$.*

(*i*) *We have*

$$T_{\psi}^{A,A,A}(X,X) = \widehat{\psi}(A) \times X^2.$$

(*ii*) *We have*

$$T_{\psi}^{A,A,A}(Y,X) = T_{\phi_1}^{A,A}(Y) \times X, \quad Y \in B(\mathbb{C}^n),$$

where

$$\phi_1(x_0,x_1) = \psi(x_0,x_1,x_1), \quad x_0, x_1 \in \mathbb{R}.$$

(*iii*) *We have*

$$T_{\psi}^{A,A,A}(X,Y) = X \times T_{\phi_2}^{A,A}(Y), \quad Y \in B(\mathbb{C}^n),$$

where

$$\phi_2(x_0,x_1) = \psi(x_0,x_0,x_1), \quad x_0, x_1 \in \mathbb{R}.$$

Proof. Let $\{\xi_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of A and let $\{\lambda_i\}_{i=1}^n$ be the associated n-tuple of eigenvalues. Since A commutes with X, it follows that the projection P_{ξ_i} commutes with X for all $1 \le i \le n$. Thus, we have that

$$\begin{aligned}
T_{\psi}^{A,A,A}(X,X) &= \sum_{i,j,k=1}^{n} \psi(\lambda_i, \lambda_k, \lambda_j) P_{\xi_i} X P_{\xi_k} X P_{\xi_j} \\
&= \sum_{i=1}^{n} \psi(\lambda_i, \lambda_i, \lambda_i) P_{\xi_i} \times X^2 \\
&= \sum_{i=1}^{n} \widehat{\psi}(\lambda_i) P_{\xi_i} \times X^2 \;=\; \widehat{\psi}(A) \times X^2,
\end{aligned}$$

which proves (*i*).

Similarly, for (ii), we have

$$\begin{aligned} T_{\psi}^{A,A,A}(Y,X) &= \sum_{i,j,k=1}^{n} \psi(\lambda_i,\lambda_k,\lambda_j) P_{\xi_i} Y P_{\xi_k} X P_{\xi_j} \\ &= \sum_{i,k=1}^{n} \psi(\lambda_i,\lambda_k,\lambda_k) P_{\xi_i} Y P_{\xi_k} \times X \\ &= \sum_{i,k=1}^{n} \phi_1(\lambda_i,\lambda_k) P_{\xi_i} Y P_{\xi_k} \times X = T_{\phi_1}^{A,A}(Y) \times X. \end{aligned}$$

The proof of (iii) repeats that of (ii). □

3.4. **Divided differences.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that f admits right and left derivatives $f'_r(x)$ and $f'_l(x)$ at each $x \in \mathbb{R}$. Assume further that f'_r, f'_l are bounded. The divided difference of the first order is defined by

$$f^{[1]}(x_0,x_1) := \begin{cases} \frac{f(x_0)-f(x_1)}{x_0-x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f'_r(x_0)+f'_l(x_0)}{2} & \text{if } x_0 = x_1 \end{cases}, \quad x_0,x_1 \in \mathbb{R}.$$

Then $f^{[1]}$ is a bounded Borel function.

Let A_0, A_1 as in Subsection 3.1. We study below the multiplier $T_{f^{[1]}}^{A_0,A_1}$ and give the formula from [1, Theorem 5.3] in the setting of matrices (see (22) below). The symbol $f^{[1]}$ and the corresponding Schur multiplier were first studied by Löwner in [18], where he noted that since

$$f(A_j)\xi_i^{(j)} = f(\lambda_i^{(j)})\xi_i^{(j)}, \quad 1 \le i \le n, \quad j = 0,1,$$

we have

$$\big\langle (f(A_0)-f(A_1))(\xi_k^{(1)}), \xi_i^{(0)} \big\rangle = f^{[1]}(\lambda_i^{(0)},\lambda_k^{(1)}) \big\langle (A_0-A_1)(\xi_k^{(1)}), \xi_i^{(0)} \big\rangle. \tag{21}$$

Formula (21) implies that

$$f(A_0) - f(A_1) = T_{f^{[1]}}^{A_0,A_1}(A_0 - A_1). \tag{22}$$

Now assume that f is a C^2-function, with a bounded second derivative f''. The divided difference of the second order is defined by

$$f^{[2]}(x_0,x_1,x_2) := \begin{cases} \frac{f^{[1]}(x_0,x_1)-f^{[1]}(x_1,x_2)}{x_0-x_2}, & \text{if } x_0 \neq x_2, \\ \frac{d}{dx_0} f^{[1]}(x_0,x_1), & \text{if } x_0 = x_2 \end{cases}, \quad x_0,x_1,x_2 \in \mathbb{R}. \tag{23}$$

Then $f^{[2]}$ is a bounded Borel function, and this function is symmetric in the three variables (x_0,x_1,x_2).

The following result may be viewed as a higher dimensional version of (22).

Theorem 15. *Let $f \in C^2(\mathbb{R})$ and $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be self-adjoint operators. Then for all $X \in B(\mathbb{C}^n)$ we have*

$$T_{f^{[1]}}^{A_0,A_2}(X) - T_{f^{[1]}}^{A_1,A_2}(X) = T_{f^{[2]}}^{A_0,A_1,A_2}(A_0 - A_1, X).$$

Proof. Let $X \in B(\mathbb{C}^n)$ and let $\psi = f^{[2]}$ and $\phi = f^{[1]}$. Setting $\psi_0, \psi_1, \tilde{\psi}_0, \tilde{\psi}_1$ as in Lemma 9 (i), (ii), we have

$$\begin{aligned} (\psi_0 - \psi_1)(x_0,x_1,x_2) &= x_0 f^{[2]}(x_0,x_1,x_2) - x_1 f^{[2]}(x_0,x_1,x_2) \\ &= f^{[1]}(x_0,x_2) - f^{[1]}(x_1,x_2) \\ &= (\tilde{\psi}_0 - \tilde{\psi}_1)(x_0,x_1,x_2). \end{aligned} \tag{24}$$

Therefore, by Lemma 9, we obtain

$$\begin{aligned}
T_{f^{[2]}}^{A_0,A_1,A_2}(A_0 - A_1, X) &= T_{f^{[2]}}^{A_0,A_1,A_2}(A_0, X) - T_{f^{[2]}}^{A_0,A_1,A_2}(A_1, X) \\
&\overset{Lem 9(i)}{=} T_{\psi_0}^{A_0,A_1,A_2}(I_n, X) - T_{\psi_1}^{A_0,A_1,A_2}(I_n, X) \\
&= T_{\psi_0 - \psi_1}^{A_0,A_1,A_2}(I_n, X) \\
&\overset{(24)}{=} T_{\tilde{\psi}_0 - \tilde{\psi}_1}^{A_0,A_1,A_2}(I_n, X) \\
&= T_{\tilde{\psi}_0}^{A_0,A_1,A_2}(I_n, X) - T_{\tilde{\psi}_1}^{A_0,A_1,A_2}(I_n, X) \\
&\overset{Lem 9(ii)}{=} T_{f^{[1]}}^{A_0,A_2}(X) - T_{f^{[1]}}^{A_1,A_2}(X).
\end{aligned}$$

□

Let $f \in C^1(\mathbb{R})$ and let $A, B \in B(\mathbb{C}^n)$ be self-adjoint operators. Then the function $t \mapsto f(A + tB)$ is differentiable and

$$\frac{d}{dt}\big(f(A + tB)\big)\Big|_{t=0} = T_{f^{[1]}}^{A,A}(B). \tag{25}$$

Indeed this follows e.g. from [14, Theorem 3.25]. This leads to the following reformulation of (3) in terms of bilinear Schur multipliers.

Theorem 16. *For any self-adjoint operators $A, B \in B(\mathbb{C}^n)$ and any $f \in C^2(\mathbb{R})$, we have*

$$f(A + B) - f(A) - \frac{d}{dt}\big(f(A + tB)\big)\Big|_{t=0} = T_{f^{[2]}}^{A+B,A,A}(B, B). \tag{26}$$

Proof. By (22), we have that

$$f(A + B) - f(A) = T_{f^{[1]}}^{A+B,A}(B).$$

Combining with (25) and applying Theorem 15, we arrive at

$$\begin{aligned}
f(A + B) - f(A) - \frac{d}{dt}\big(f(A + tB)\big)\Big|_{t=0} &= T_{f^{[1]}}^{A+B,A}(B) - T_{f^{[1]}}^{A,A}(B) \\
&= T_{f^{[2]}}^{A+B,A,A}(B, B).
\end{aligned}$$

□

4. Finite-dimensional construction

In this section we establish various estimates concerning finite dimensional operators. The symbol const will stand for uniform positive constants, not depending on the dimension.

Consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(x) = |x|, \qquad x \in \mathbb{R}.$$

The definition of $f_0^{[1]}$ given in Subsection 3.4 applies to this function.

The following result is proved in [9, Theorem 13].

Theorem 17. *For all $n \in \mathbb{N}$ there exist self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, 0 is an eigenvalue of A_n, and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq \text{const } \log n \|B_n\|_1. \tag{27}$$

Remark 18. The operator A_n constructed in [9] is a diagonal operator defined on $\mathbb{C}^{2n}$ and 0 is not an eigenvalue of A_n. By changing the dimension from $2n$ to $2n+1$ and adding a zero on the diagonal, one obtains the operator A_n in Theorem 17, with 0 in the spectrum.

Corollary 19. *For all $n \geq 1$, there exist self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, and*

$$\left\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\| \geq \text{const } \log n.$$

Proof. Take $A_n, B_n \in B(\mathbb{C}^{2n+1})$ as in Theorem 17. By (22), we have that

$$T^{A_n+B_n,A_n}_{f_0^{[1]}}(B_n) = f_0(A_n+B_n) - f_0(A_n).$$

By Theorem 17, we have that

$$\|T^{A_n+B_n,A_n}_{f_0^{[1]}}(B_n)\|_1 = \|f_0(A_n+B_n) - f_0(A_n)\|_1 \geq \text{const } \log n \|B_n\|_1.$$

Therefore,

$$\left\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^1_{2n+1} \to \mathcal{S}^1_{2n+1}\right\| \geq \text{const } \log n.$$

Since the operator $T^{A_n+B_n,A_n}_{f_0^{[1]}}$ is a Schur multiplier, we obtain that

$$\left\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\right\| \geq \text{const } \log n.$$

□

Consider the function $g_0 : \mathbb{R} \to \mathbb{R}$ given by

$$g_0(x) = x|x| = xf_0(x), \qquad x \in \mathbb{R}.$$

Although g_0 is not a C^2-function, one may define $g_0^{[2]}(x_0, x_1, x_2)$ by (23) whenever x_0, x_1, x_2 are not equal. Let us define

$$\psi_0(x_0,x_1,x_2) := \begin{cases} g_0^{[2]}(x_0,x_1,x_2), & \text{if } \ x_0 \neq x_1 \ \text{ or } \ x_1 \neq x_2 \\ 2, & x_0 = x_1 = x_2 > 0 \\ -2, & x_0 = x_1 = x_2 < 0 \\ 0, & \text{if } \ x_0 = x_1 = x_2 \end{cases}.$$

The function $\psi_0 \colon \mathbb{R}^3 \to \mathbb{C}$ is a bounded Borel function.

The following lemma relates the linear Schur multiplier for $f_0^{[1]}$ and the bilinear Schur multiplier for ψ_0.

Lemma 20. *For self-adjoint operators $A_n, B_n \in B(\mathbb{C}^n)$ such that 0 belongs to the spectrum of A_n, the inequality*

$$\left\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\right\| \geq \left\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\right\| \tag{28}$$

holds.

Proof. Let $\{\mu_k\}_{k=1}^n$ be the sequence of eigenvalues of the operator A_n. For simplicity, we assume that $\mu_1 = 0$.

By formulas (16) and (19) and by Theorem 6, we have that

$$\left\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\right\| = \max_{1\leq k\leq n} \|T^{A_n+B_n,A_n}_{\varphi_k} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\|,$$

where

$$\varphi_k(x_0,x_1) := \psi_0(x_0,\mu_k,x_1), \quad x_0,x_1 \in \mathbb{R}, \ 1 \leq k \leq n.$$

In particular, we have

$$\left\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\right\| \geq \|T^{A_n+B_n,A_n}_{\varphi_1} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\|.$$

It therefore suffices to check that

$$\varphi_1 = f_0^{[1]}. \tag{29}$$

It follows from the definitions that $\varphi_1(0,0) = \psi_0(0,0,0) = 0 = f_0^{[1]}(0,0)$.

Consider now $(x_0, x_1) \in \mathbb{R}^2$ such that $x_0 \neq 0$ or $x_1 \neq 0$. In that case, we have

$$\varphi_1(x_0, x_1) = g_0^{[2]}(x_0, 0, x_1).$$

If $x_0, x_1, 0$ are mutually distinct, then

$$\begin{aligned} g_0^{[2]}(x_0, 0, x_1) &= \frac{g_0^{[1]}(x_0, 0) - g_0^{[1]}(0, x_1)}{x_0 - x_1} = \frac{\frac{x_0 f_0(x_0) - 0}{x_0 - 0} - \frac{0 - x_1 f_0(x_1)}{0 - x_1}}{x_0 - x_1} \\ &= \frac{f_0(x_0) - f_0(x_1)}{x_0 - x_1} = f_0^{[1]}(x_0, x_1). \end{aligned}$$

If $x_0 = 0$ and $x_1 \neq 0$, then

$$\begin{aligned} g_0^{[2]}(0, 0, x_1) &= \frac{g_0^{[1]}(0, 0) - g_0^{[1]}(0, x_1)}{x_0 - x_1} = \frac{g_0'(0) - \frac{0 - x_1 f_0(x_1)}{0 - x_1}}{0 - x_1} \\ &= \frac{f_0(x_1)}{x_1} = f_0^{[1]}(0, x_1). \end{aligned}$$

The argument is similar, when $x_0 \neq 0$ and $x_1 = 0$.

Assume now that $x_0 = x_1 \neq 0$. Then we have

$$\begin{aligned} g_0^{[2]}(x_0, 0, x_0) &= \frac{d}{dx} g_0^{[1]}(x, 0)\Big|_{x = x_0} = \frac{d}{dx}\Big(\frac{x f_0(x) - 0}{x - 0}\Big)\Big|_{x = x_0} \\ &= f_0'(x_0) = f_0^{[1]}(x_0, x_0). \end{aligned}$$

This completes the proof of (29) and we obtain (28). $\square$

The following is a straightforward consequence of Corollary 19 and Lemma 20.

Corollary 21. *For every $n \geq 1$ there exist self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, and*

$$\left\| T_{\psi_0}^{A_n + B_n, A_n, A_n} : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1} \right\| \geq \text{const } \log n.$$

We assume below that $n \geq 1$ is fixed and that A_n, B_n are given by Corollary 21. The purpose of the series of lemmas 22-27 below is to prove Lemma 28, which is the final step in the finite-dimensional resolution of Peller's problem. The following result follows immediately from Corollary 21.

Lemma 22. *There are operators $X_n, Y_n \in B(\mathbb{C}^{2n+1})$ with $\|X_n\|_2 = \|Y_n\|_2 = 1$, such that*

$$\left\| T_{\psi_0}^{A_n + B_n, A_n, A_n}(X_n, Y_n) \right\|_1 \geq \text{const } \log n.$$

Let us denote

$$H_n := \begin{pmatrix} A_n + B_n & 0 \\ 0 & A_n \end{pmatrix} \tag{30}$$

and consider the operator

$$T_1 := T_{\psi_0}^{H_n, H_n, H_n} : \mathcal{S}^2_{4n+2} \times \mathcal{S}^2_{4n+2} \to \mathcal{S}^1_{4n+2}.$$

Lemma 23. *There are operators $\tilde{X}_n, \tilde{Y}_n \in B(\mathbb{C}^{4n+2})$ with $\|\tilde{X}_n\|_2 = \|\tilde{Y}_n\|_2 = 1$, such that*

$$\left\| T_1(\tilde{X}_n, \tilde{Y}_n) \right\|_1 \geq \text{const } \log n.$$

Proof. Take

$$\tilde{X}_n := \begin{pmatrix} 0 & X_n \\ 0_{2n+1} & 0 \end{pmatrix}, \quad \tilde{Y}_n := \begin{pmatrix} 0_{2n+1} & 0 \\ 0 & Y_n \end{pmatrix},$$

where X_n, Y_n are operators from Lemma 22 and 0_{2n+1} is the null element of $B(\mathbb{C}^{2n+1})$. Clearly, $\|\tilde{X}_n\|_2 = \|X_n\|_2 = 1$ and $\|\tilde{Y}_n\|_2 = \|Y_n\|_2 = 1$. It follows from Lemma 11 and the fact that $A_n + B_n$ have the same spectra that

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \begin{pmatrix} 0 & T_{\psi_0}^{A_n+B_n, A_n, A_n}(X_n, Y_n) \\ 0_{2n+1} & 0 \end{pmatrix}.$$

Therefore, by Lemma 22,

$$\big\|T_1(\tilde{X}_n, \tilde{Y}_n)\big\|_1 = \big\|T_{\psi_0}^{A_n+B_n, A_n, A_n}(X_n, Y_n)\big\|_1 \geq \text{const } \log n.$$

□

Lemma 24. *There is an operator $S_n \in B(\mathbb{C}^{4n+2})$ with $\|S_n\|_2 \leq 1$ such that*

$$\big\|T_1(S_n, S_n^*)\big\|_1 \geq \text{const } \log n.$$

Proof. Take the operators $\tilde{X}_n, \tilde{Y}_n \in B(\mathbb{C}^{4n+2})$ as in Lemma 23. By the polarization identity

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \frac{1}{4}\sum_{k=0}^{3} i^k T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*),$$

we have that

$$\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \leq \max_{0\leq k\leq 3} \|T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*)\|_1.$$

Taking k_0 such that

$$\|T_1((\tilde{X}_n + i^{k_0}\tilde{Y}_n^*), (\tilde{X}_n + i^{k_0}\tilde{Y}_n^*)^*)\|_1 = \max_{0\leq k\leq 3} \|T_1((\tilde{X}_n + i^k\tilde{Y}_n^*), (\tilde{X}_n + i^k\tilde{Y}_n^*)^*)\|_1,$$

we set

$$S_n := \frac{1}{2}(\tilde{X}_n + i^{k_0}\tilde{Y}_n^*).$$

Thus, by Lemma 23, we have

$$\big\|T_1(S_n, S_n^*)\big\|_1 \geq \frac{1}{4}\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \geq \text{const } \log n$$

and

$$\|S_n\|_2 \leq \frac{1}{2}(\|\tilde{X}_n\|_2 + \|\tilde{Y}_n\|_2) = 1.$$

□

Let us denote

$$\tilde{H}_n := \begin{pmatrix} H_n & 0 \\ 0 & H_n \end{pmatrix} = \begin{pmatrix} A_n+B_n & 0 & 0 & 0 \\ 0 & A_n & 0 & 0 \\ 0 & 0 & A_n+B_n & 0 \\ 0 & 0 & 0 & A_n \end{pmatrix}, \quad n \geq 1, \tag{31}$$

and consider the operator

$$T_2 := T_{\psi_0}^{\tilde{H}_n, \tilde{H}_n, \tilde{H}_n} : \mathcal{S}^2_{8n+4} \times \mathcal{S}^2_{8n+4} \to \mathcal{S}^1_{8n+4}.$$

Lemma 25. *There is a self-adjoint operator $Z_n \in B(\mathbb{C}^{8n+4})$ with $\|Z_n\|_2 \leq 1$ such that*

$$\big\|T_2(Z_n, Z_n)\big\|_1 \geq \text{const } \log n.$$

Proof. Consider the operator S_n from Lemma 24. Setting

$$Z_n := \frac{1}{2}\begin{pmatrix} 0 & S_n \\ S_n^* & 0 \end{pmatrix},$$

we have $\|Z_n\|_2 = \frac{1}{2}(\|S_n\|_2 + \|S_n^*\|_2) \leq 1$ and by Lemma 10,

$$T_2(Z_n, Z_n) = \frac{1}{4}\begin{pmatrix} T_1(S_n, S_n^*) & 0 \\ 0 & T_1(S_n^*, S_n) \end{pmatrix}.$$

Therefore, by Lemma 24, we arrive at

$$\begin{aligned} \left\|T_2(Z_n, Z_n)\right\|_1 &= \frac{1}{4}\left(\left\|T_1(S_n, S_n^*)\right\|_1 + \left\|T_1(S_n^*, S_n)\right\|_1\right) \\ &\geq \frac{1}{4}\left\|T_1(S_n, S_n^*)\right\|_1 \geq \text{const } \log n. \end{aligned}$$

□

The following decomposition principle is of independent interest. In this statement we use the notation $[H, F] = HF - FH$ for the commutator of H and F.

Lemma 26. *For any self-adjoint operators $Z, H \in B(\mathbb{C}^n)$, there are self-adjoint operators $F, G \in B(\mathbb{C}^n)$ such that*

$$Z = G + i[H, F],$$

the matrix G commutes with H, and

$$\|[H, F]\|_2 \leq 2\, \|Z\|_2 \quad \text{and} \quad \|G\|_2 \leq \|Z\|_2\,.$$

Proof. Let

$$h_1, h_2, \dots, h_m$$

be the pairwise distinct eigenvalues of the operator H and let

$$E_1, E_2, \dots, E_m$$

be the associated spectral projections, so that

$$H = \sum_{j=1}^{m} h_j E_j.$$

We set

$$G = \sum_{j=1}^{m} E_j Z E_j \qquad \text{and} \qquad F = i\sum_{\substack{j=1 \\ j\neq k}}^{m} (h_k - h_j)^{-1}\, E_j Z E_k.$$

Since

$$HE_j = h_j E_j,$$

we have

$$[H, E_j Z E_k] = \ H \times E_j Z E_k - \ E_j Z E_k \times H = (h_j - h_k) \times E_j Z E_k.$$

Consequently,

$$i[H, F] = \sum_{\substack{j=1 \\ j\neq k}}^{m} E_j Z E_k$$

and hence

$$G + i[H, F] = Z.$$

Further F, G are self-adjoint and it is clear that $[G, H] = 0$. Hence the first two claims of the lemma are proved.

Now take

$$U_t = \sum_{j=1}^{m} e^{ijt}\, E_j, \quad t \in [-\pi, \pi].$$

Then

$$\int_{-\pi}^{\pi} U_t Z U_t^* \,\frac{\mathrm{d}t}{2\pi} = \sum_{j,k=1}^{m} E_j Z E_k \int_{-\pi}^{\pi} e^{i(j-k)\,t}\,\frac{\mathrm{d}t}{2\pi} = \sum_{j=1}^{m} E_j Z E_j \;= G.$$

Since U_t is unitary, we deduce that

$$\|G\|_2 \leq \int_{-\pi}^{\pi} \|U_t Z U_t^*\|_2 \,\frac{\mathrm{d}t}{2\pi} \leq \|Z\|_2\,.$$

Moreover writing

$$i[H, F] = Z - G$$

we deduce that

$$\|[H, F]\|_2 \leq 2\,\|Z\|_2\,.$$

□

Lemma 27. *There is a self-adjoint operator $F_n \in B(\mathbb{C}^{8n+4})$ such that $\|[\tilde{H}_n, F_n]\|_2 \leq 2$ and*

$$\big\|T_2\big(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n]\big)\big\|_1 \geq \text{const}\ \log n - 10.$$

Proof. Take the operator Z_n in $B(\mathbb{C}^{8n+4})$ given by Lemma 25. By Lemma 26, we may choose self-adjoint operators F_n and G_n from $B(\mathbb{C}^{8n+4})$ such that

$$Z_n = G_n + i[\tilde{H}_n, F_n], \quad [G_n, \tilde{H}_n] = 0,$$

and

$$\|[\tilde{H}_n, F_n]\|_2 \leq 2\,\|Z_n\|_2\,, \qquad \|G_n\|_2 \leq \|Z_n\|_2\,. \tag{32}$$

We compute

$$\begin{aligned} T_2(Z_n, Z_n) &= T_2\Big(G_n + i[\tilde{H}_n, F_n],\, G_n + i[\tilde{H}_n, F_n]\Big) \\ &= T_2\Big(G_n,\, G_n\Big) \\ &+ T_2\Big(G_n,\, i[\tilde{H}_n, F_n]\Big) \\ &+ T_2\Big(i[\tilde{H}_n, F_n],\, G_n\Big) \\ &+ T_2\Big(i[\tilde{H}_n, F_n],\, i[\tilde{H}_n, F_n]\Big). \end{aligned} \tag{33}$$

We shall estimate the first three summands above. The operator G_n commutes with $\tilde{H}_n$ hence by the first part of Lemma 14,

$$T_2(G_n, G_n) = \widehat{\psi_0}(\tilde{H}_n) \times G_n^2.$$

Furthermore $\widehat{\psi_0}(x) = 2$ if $x > 0$, $\widehat{\psi_0}(x) = -2$ if $x < 0$ and $\widehat{\psi_0}(0) = 0$. Hence

$$\|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \leq 2.$$

This implies that

$$\big\|T_2(G_n, G_n)\big\|_1 \leq \|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \|G_n\|_2^2 \leq 2\|Z_n\|_2^2 \leq 2.$$

Next applying the second and third part of Lemma 14, we obtain

$$T_2\Big(i[\tilde{H}_n, F_n],\, G_n\Big) = iT_{\phi_1}^{\tilde{H}_n, \tilde{H}_n}\Big([\tilde{H}_n, F_n]\Big) \times G_n$$

and

$$T_2\Big(G_n, i[\tilde{H}_n, F_n]\Big) = i\; G_n \times T_{\phi_2}^{\tilde{H}_n, \tilde{H}_n}\Big([\tilde{H}_n, F_n]\Big),$$

where

$$\phi_1(x_0,x_1) = \psi_0(x_0,x_1,x_1) \quad \text{and} \quad \phi_2(x_0,x_1) = \psi_0(x_0,x_0,x_1), \quad x_0,x_1 \in \mathbb{R}.$$

Observe that by the Mean Value Theorem for divided differences (see e.g. [8]), we have $\|\psi_0\|_\infty \le 2$. Hence $\|\phi_1\|_\infty \le 2$ and $\|\phi_2\|_\infty \le 2$, which implies

$$\begin{aligned}\left\|T_{\phi_1}^{\tilde{H}_n,\tilde{H}_n}\Big([\tilde{H}_n,F_n]\Big)\times G_n\right\|_1 &\le \left\|T_{\phi_1}^{\tilde{H}_n,\tilde{H}_n}\Big([\tilde{H}_n,F_n]\Big)\right\|_2\|G_n\|_2\\ &\le \|\phi_1\|_\infty\|[\tilde{H}_n,F_n]\|_2\|G_n\|_2\\ &\le 2\|\phi_1\|_\infty\|Z_n\|_2^2 \le 4\end{aligned}$$

by (32) and Lemma 25. Similarly,

$$\left\|G_n \times T_{\phi_2}^{\tilde{H}_n,\tilde{H}_n}\Big([\tilde{H}_n,F_n]\Big)\right\|_1 \le 4.$$

Combining the preceding estimates with (33), we arrive at

$$\|T_2(Z_n,Z_n)\|_1 \le 10 + \left\|T_2\Big(i[\tilde{H}_n,F_n],\, i[\tilde{H}_n,F_n]\Big)\right\|_1.$$

Applying Lemma 25, we deduce the result. $\square$

Lemma 28. *There exists a C^2-function g with a bounded second derivative and there exists $N\in\mathbb{N}$ such that for any sequence $\{\alpha_n\}_{n\ge N}$ of positive real numbers there is a sequence of operators $\tilde{B}_n \in B(\mathbb{C}^{8n+4})$ such that $\|\tilde{B}_n\|_2 \le 4\alpha_n$, for all $n\ge N$, and*

$$\|T_{g^{[2]}}^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}(\tilde{B}_n,\tilde{B}_n)\|_1 \ge \mathrm{const}\,\alpha_n^2\ \log n, \quad n\ge N.$$

Proof. Changing the constant 'const' in Lemma 27 by half of its value, we can change the estimate from that statement into

$$\left\|T_2\big(i[\tilde{H}_n,F_n], i[\tilde{H}_n,F_n]\big)\right\|_1 \ge \mathrm{const}\ \log n, \quad n\ge N, \tag{34}$$

for sufficiently large $N\in\mathbb{N}$.

Take an arbitrary sequence $\{\alpha_n\}_{n\ge N}$ of positive real numbers, take the operator F_n from Lemma 27 and denote

$$\tilde{F}_n := \alpha_n F_n.$$

For any $t>0$, consider

$$\gamma_t(\tilde{H}_n) = e^{it\tilde{F}_n}\tilde{H}_n e^{-it\tilde{F}_n}, \quad \text{and} \quad V_{n,t} := \frac{\gamma_t(\tilde{H}_n)-\tilde{H}_n}{t}.$$

On the one hand, it follows from the identity $\frac{d}{dt}(e^{it\tilde{F}_n})|_{t=0} = i\tilde{F}_n$ that

$$V_{n,t} \longrightarrow i[\tilde{F}_n,\tilde{H}_n], \quad t\to +0.$$

It therefore follows from Lemma 27 that there is $t_1>0$ such that

$$\|V_{n,t}\|_2 \le 2\|[\tilde{F}_n,\tilde{H}_n]\|_2 = 2\alpha_n\|[F_n,\tilde{H}_n]\|_2 \le 4\alpha_n \tag{35}$$

for all $t\le t_1$. On the other hand,

$$\tilde{H}_n + t\,V_{n,t} = \gamma_t(\tilde{H}_n) \longrightarrow \tilde{H}_n, \quad t\to+0. \tag{36}$$

Take a C^2-function g such that $g(x)=g_0(x)=x|x|$ for $|x|>1$ and $g^{(j)}(0)=0$, $j=0,1,2$. Denote

$$g_t(x_0,x_1,x_2) := g^{[2]}\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big), \quad t>0, \quad x_0,x_1,x_2\in\mathbb{R}.$$

We claim that

$$\lim_{t\to+0} g_t(x_0,x_1,x_2) = \psi_0(x_0,x_1,x_2), \quad x_0,x_1,x_2\in\mathbb{R}. \tag{37}$$

To prove this claim, we first observe, using the definition of g_0, that

$$\psi_0\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t},\Big) = \psi_0(x_0,x_1,x_2), \quad x_0,x_1,x_2\in\mathbb{R},\ t>0. \tag{38}$$

Next we note that for any $x\in\mathbb{R}$,

$$g\Big(\frac{x}{t}\Big) = g_0\Big(\frac{x}{t}\Big) \quad \text{and} \quad g'\Big(\frac{x}{t}\Big) = g_0'\Big(\frac{x}{t}\Big)$$

for $t>0$ small enough. For $x=0$, this follows from the fact that by assumption, $g(0)=g'(0)=0$. From these properties, we deduce that for any $x_0,x_1\in\mathbb{R}$,

$$g^{[1]}\Big(\frac{x_0}{t},\frac{x_1}{t}\Big) = g_0^{[1]}\Big(\frac{x_0}{t},\frac{x_1}{t}\Big)$$

for $t>0$ small enough.

In turn, this implies that if $x_0\neq x_1$ or $x_1\neq x_2$, then

$$g^{[2]}\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big) = g_0^{[2]}\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big)$$

for $t>0$ small enough. According to (38), this implies that

$$g^{[2]}\Big(\frac{x_0}{t},\frac{x_1}{t},\frac{x_2}{t}\Big) = \psi_0(x_0,x_1,x_2)$$

for $t>0$ small enough.

Consider now the case when $x_0=x_1=x_2$. For any $t>0$, we have

$$g^{[2]}\Big(\frac{x_0}{t},\frac{x_0}{t},\frac{x_0}{t}\Big) = g_0''\Big(\frac{x_0}{t}\Big).$$

If $x_0>0$, then $g_0''\big(\frac{x_0}{t}\big)=2$ for $t>0$ small enough, and if $x_0<0$, then $g_0''\big(\frac{x_0}{t}\big)=-2$ for $t>0$ small enough. Furthermore, $g_0''(0)=0$ by assumption. Hence

$$g^{[2]}\Big(\frac{x_0}{t},\frac{x_0}{t},\frac{x_0}{t}\Big) = \psi_0(x_0,x_0,x_0)$$

for $t>0$ small enough. This completes the proof of (37).

Applying subsequently Lemma 12 with $a=\frac{1}{t}$, property (36) and Lemma 13, we obtain that

$$\begin{aligned} T_{g^{[2]}}^{\frac{1}{t}\tilde H_n+V_{n,t},\frac{1}{t}\tilde H_n,\frac{1}{t}\tilde H_n}(V_{n,t},V_{n,t}) &= T_{g_t}^{\tilde H_n+tV_{n,t},\tilde H_n,\tilde H_n}(V_{n,t},V_{n,t}) \\ &\longrightarrow T_2\big(i[\tilde F_n,\tilde H_n], i[\tilde F_n,\tilde H_n]\big) \end{aligned}$$

when $t\to+0$. Furthermore,

$$T_2\big(i[\tilde F_n,\tilde H_n], i[\tilde F_n,\tilde H_n]\big) = \alpha_n^2 T_2\big(i[F_n,\tilde H_n], i[F_n,\tilde H_n]\big).$$

By (34), there is $t_2>0$ such that

$$\big\|T_{g^{[2]}}^{\frac{1}{t}\tilde H_n+V_{n,t},\frac{1}{t}\tilde H_n,\frac{1}{t}\tilde H_n}(V_{n,t},V_{n,t})\big\|_1 \geq \text{const}\ \alpha_n^2\ \log n$$

for all $t\leq t_2$. Taking $t_n=\min\{t_1,t_2\}$, and setting

$$\tilde A_n := \frac{1}{t_n}\tilde H_n, \quad \tilde B_n := V_{n,t_n},$$

we obtain that $\|\tilde B_n\|_2\leq 4\alpha_n$ (see (35)) and

$$\|T_{g^{[2]}}^{\tilde A_n+\tilde B_n,\tilde A_n,\tilde A_n}(\tilde B_n,\tilde B_n)\|_1 \geq \text{const}\,\alpha_n^2\ \log n,$$

for all $n\geq N$. □

5. Answering Peller's problem

Let $\{\mathcal{H}_n\}_{n=1}^{\infty}$ be a sequence of finite dimensional Hilbert spaces and consider their Hilbertian direct sum

$$\mathcal{H} = \overset{2}{\oplus}_{n\geq 1}\mathcal{H}_n.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of self-adjoint operators, with $A_n \in B(\mathcal{H}_n)$. Let A denote their direct sum (notation $A = \oplus_{n=1}^{\infty} A_n$). Namely A is defined on the domain

$$D(A) = \Big\{\{\xi_n\}_{n=1}^{\infty} \in \mathcal{H} \,:\, \sum_{n=1}^{\infty} \|A_n(\xi_n)\|^2 < \infty\Big\},$$

by setting $A(\xi) = \{A_n(\xi_n)\}_{n=1}^{\infty}$ for any $\xi = \{\xi_n\}_{n=1}^{\infty}$ in $D(A)$. Then A is a self-adjoint (possibly unbounded) operator on $\mathcal{H}$.

Likewise we let $\{B_n\}_{n=1}^{\infty}$ be a sequence of self-adjoint operators, with $B_n \in \mathcal{S}^2(\mathcal{H}_n)$, and we set $B = \oplus_{n=1}^{\infty} B_n$. Assume further that $\sum_{n=1}^{\infty} \|B_n\|_2^2 < \infty$. Then $B \in \mathcal{S}^2(\mathcal{H})$ and

$$\|B\|_2^2 = \sum_{n=1}^{\infty} \|B_n\|_2^2. \tag{39}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^2-function with a bounded second derivative. Then $f^{[2]}$ is bounded, with $\|f^{[2]}\|_\infty = \|f''\|_\infty$. Hence according to Theorem 16 and Lemma 3, we have

$$\Big\| f(A_n + B_n) - f(A_n) - \frac{d}{dt}\Big(f(A_n + tB_n)\Big)\Big|_{t=0}\Big\|_2 \leq \|f''\|_\infty \|B_n\|_2^2.$$

We deduce that

$$\sum_{n=1}^{\infty} \Big\| f(A_n + B_n) - f(A_n) - \frac{d}{dt}\Big(f(A_n + tB_n)\Big)\Big|_{t=0}\Big\|_2^2 \leq \|f''\|_\infty^2 \Big(\sum_{n=1} \|B_n\|_2^2\Big)^2 < \infty.$$

Then we may define

$$f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0} := \bigoplus_{n=1}^{\infty} \Big(f(A_n + B_n) - f(A_n) - \frac{d}{dt}\Big(f(A_n + tB_n)\Big)\Big|_{t=0}\Big),$$

which is an element of $\mathcal{S}^2(\mathcal{H})$.

We note that the above construction can be carried out as well in the case when the $\mathcal{H}_n$'s are infinite dimensional, provided that each A_n is a bounded operator.

The following theorem answers Peller's problem (5) in negative.

Theorem 29. *There exists a function $f \in C^2(\mathbb{R})$ with a bounded second derivative, a self-adjoint operator A on $\mathcal{H}$ and a self-adjoint $B \in \mathcal{S}^2(\mathcal{H})$ as above such that*

$$f(A+B) - f(A) - \frac{d}{dt}\Big(f(A+tB)\Big)\Big|_{t=0} \notin \mathcal{S}^1.$$

Proof. Take the integer $N \in \mathbb{N}$, the operators $\tilde{A}_n$, $\tilde{B}_n$ and the function g from Lemma 28, applied with the sequence $\{\alpha_n\}_{n\geq N}$ defined by

$$\alpha_n = \frac{1}{\sqrt{n\ \log^{3/2} n}}.$$

Let $\mathcal{H}_n = \ell^2_{8n+4}$ and let $\mathcal{H} = \overset{2}{\oplus}_{n\geq N}\mathcal{H}_n$. Then let $A = \oplus_{n=N}^{\infty} A_n$ and $B = \oplus_{n=N}^{\infty} B_n$ be the corresponding direct sums. Then the self-adjoint operator B belongs to $S^2(\mathcal{H})$. Indeed, it follows from (39) and Lemma 28 that

$$\|B\|_2^2 = \sum_{n=N}^{\infty} \|\tilde{B}_n\|_2^2 \leq 16 \sum_{n=N}^{\infty} \alpha_n^2 = \sum_{n=N}^{\infty} \frac{16}{n\ \log^{3/2} n} < \infty.$$

On the other hand, by (26) and Lemma 28, we have

$$\begin{aligned}
\Big\|g(A+B)-g(A) &- \frac{d}{dt}\big(g(A+tB)\big)\Big|_{t=0}\Big\|_1 \\
&= \sum_{n=N}^{\infty} \Big\|g(\tilde{A}_n+\tilde{B}_n) - g(\tilde{A}_n) - \frac{d}{dt}\big(g(\tilde{A}+t\tilde{B}_n)\big)\Big|_{t=0}\Big\|_1 \\
&= \sum_{n=N}^{\infty} \Big\|T_{g^{[2]}}^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}\Big(\tilde{B}_n,\ \tilde{B}_n\Big)\Big\|_1 \\
&\geq \text{const} \sum_{n=N}^{\infty} \alpha_n^2\ \log n \\
&= \text{const} \sum_{n=N}^{\infty} \frac{1}{n\ \log^{1/2} n} = \infty.
\end{aligned}$$

□

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